Charge & Current densities :-

Klein - Gordon eq^n for a free particle is given by :-

$$\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi = 0 \quad (10)$$

taking complex conjugate, we get

$$\nabla^2 \psi^* - \frac{1}{c^2}\frac{\partial^2 \psi^*}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi^* = 0 \quad (11)$$

multiply eq^n (10) & (11) by ψ^* and ψ respectively, we get

$$\psi^*\nabla^2\psi - \frac{1}{c^2}\psi^*\frac{\partial^2 \psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi^*\psi = 0 \quad (12)$$

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$$\psi \nabla^2 \psi^* - \frac{1}{c^2} \psi \frac{\partial^2 \psi^*}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \psi^* \psi = 0 \qquad (13)$$

Subtracting (13) from (12), we get

$$\Rightarrow \psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* - \frac{1}{c^2}\left[\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2}\right] = 0$$

$$\Rightarrow \nabla \cdot \left[\psi^* \nabla \psi - \psi \nabla \psi^*\right] - \frac{1}{c^2} \frac{\partial}{\partial t}\left[\psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi\right] = 0$$

multiply throughout by $\frac{\hbar}{2im}$, we get

$$\nabla \cdot \left[\frac{\hbar}{2im}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)\right] + \frac{\partial}{\partial t}\left[\frac{\hbar}{2imc^2}\left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right)\right] = 0 \qquad (14)$$

Substituting,

Probability density or charge density

$$P(r, t) = \frac{\hbar}{2imc^2}\left[\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right] \quad \text{(a)}$$

current density

$$\text{and } S(r, t) = \frac{\hbar}{2im}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right] \quad \text{(b)} \qquad (15)$$

eq^n (15) becomes :-

$$\boxed{\nabla \cdot S(r,t) + \frac{\partial \rho}{\partial t}(r,t) = 0} \quad (16)$$

This is the eq^n of continuity.
This shows clearly that the probability of finding the particle in any system remains conserved.

Here,

1. The current density S has the same form as in Non-relativistic case.

2. But the probability density or charge density ρ is not same as in Non-relativistic case. $\rho = \psi^* \psi$, due to following reasons :-

$$\rho(r,t) = \frac{\hbar}{2imc^2}\left[\psi\frac{\partial \psi^*}{\partial t} - \psi^*\frac{\partial \psi}{\partial t}\right]$$

$$P(r,t) = \frac{1}{2mc^2}\left[\left(-i\hbar\frac{\partial \psi^*}{\partial t}\right)\psi + \psi^*\left(-i\hbar\frac{\partial \psi}{\partial t}\right)\right] \quad \text{--- (17)}$$

$$P(r,t) = \frac{1}{2mc^2}\left[(E\psi^*)\psi + \psi^*(E\psi)\right]$$

$$= \frac{1}{2mc^2}\left[2E\psi^*\psi\right]$$

$$\boxed{P(r,t) = \frac{E}{mc^2}\left[\psi^*\psi\right]} \quad \text{--- (18)}$$

where $E = \pm\sqrt{(p^2c^2 + m^2c^4)}$